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A Virial Theorem for the Kinetic Energy of a Heavy Quark inside Hadrons

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Abstract

The formalism of the heavy quark effective theory is used to derive the field-theory analog of the virial theorem, which relates the matrix element of the kinetic energy of a heavy quark inside a hadron to a matrix element of the gluon field strength tensor. The existing QCD sum rule calculations of the kinetic energy are not consistent with this theorem.

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1 Introduction

The development of the heavy quark effective theory (HQET) has led to much progress in the theoretical understanding of the properties of hadrons, in which a heavy quark Q interacts with light degrees of freedom predominantly by the exchange of soft gluons [1, 2, 3, 4, 5]. It allows one to construct a systematic expansion of hadronic parameters (such as masses, decay constants, or transition form factors) in powers of $1/m_Q$. To leading order, the effective theory is explicitly invariant under a spin-flavor symmetry, which relates states containing a heavy quark of different spin or flavor, but with the same velocity [6, 7]. This symmetry imposes important constraints on the hadronic matrix elements of heavy quark current operators, which play a central role in the description of both exclusive and inclusive weak decays of heavy mesons and baryons. For example, in the heavy quark limit the normalization of the amplitude for the exclusive semileptonic decay $\bar{B} \rightarrow D^* \ell \bar{\nu}$ can be predicted at the kinematic point where the two heavy mesons have the same velocity, and there are no corrections of first order in $1/m_c$ or $1/m_b$ [8]. This process is thus ideal for a reliable determination of the Kobayashi-Maskawa matrix element V_{cb} [9]. For inclusive semileptonic decays of B mesons, one can prove that to leading order in $1/m_b$ the decay rates and spectra agree with the parton model prediction, and the leading nonperturbative corrections are of order $1/m_b^2$ [10, 11, 12, 13, 14].

In HQET, a heavy quark bound inside a hadron moving with four-velocity v is represented by a velocity-dependent field $h_v(x)$, which is related to the conventional spinor field $Q(x)$ by a phase redefinition [2]. When the total momentum is written as $p_Q = m_Q v + k$, the field h_v carries the residual momentum k , which results from soft interactions of the heavy quark with light degrees of freedom and is typically of order Λ_{QCD} . Moreover, the heavy quark field in the effective theory is subject to the constraint $\not{v} h_v = h_v$, which projects out the heavy quark components of the spinor. The antiquark components are integrated out to obtain the effective Lagrangian [1, 2, 3]

$$\begin{aligned} \mathcal{L}_{\text{HQET}} = & \bar{h}_v i v \cdot D h_v + \frac{1}{2m_Q} \bar{h}_v (iD_\perp)^2 h_v \\ & + C(m_Q) \frac{g_s}{4m_Q} \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v + \mathcal{O}(1/m_Q^2), \end{aligned} \quad (1)$$

where $D^\mu = \partial^\mu - i g_s A^\mu$ is the covariant derivative, $D_\perp^\mu = D^\mu - v^\mu v \cdot D$ contains

its components perpendicular to the hadron velocity, and $G^{\mu\nu}$ is the gluon field strength tensor defined by $[iD^\mu, iD^\nu] = ig_s G^{\mu\nu}$. The physical origin of the operators appearing at order $1/m_Q$ becomes transparent in the hadron's rest frame. There $(iD_\perp)^2 = \vec{D}^2$, and the first operator corresponds to the kinetic energy resulting from the residual motion of the heavy quark. The second operator describes the interaction of the heavy quark spin with the chromo-magnetic gluon field. The coefficient $C(m_Q)$ of the chromo-magnetic operator results from renormalization-group effects and contains a logarithmic dependence on the heavy quark mass [15]. The coefficient of the kinetic operator is not renormalized [16].

In order to construct a systematic $1/m_Q$ expansion, one works with the eigenstates of the leading-order term in the effective Lagrangian (1), supplemented by the standard QCD Lagrangian for the light quarks and gluons, and treats the $1/m_Q$ corrections as perturbations. The states of HQET are thus different from the physical states of the full theory. They correspond to the would-be hadrons composed of an infinitely heavy quark and light degrees of freedom. Because of the spin symmetry of the effective theory, states which are related to each other by a spin flip of the heavy quark are degenerate in mass. For instance, the ground-state pseudoscalar (P) and vector (V) mesons form a doublet under the spin symmetry. Only beyond the leading order in $1/m_Q$, a small mass splitting is induced by the presence of the spin-symmetry breaking chromo-magnetic interaction in (1). In fact, the masses of the physical states obey an expansion

$$M = m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} - d_M \frac{\lambda_2(m_Q)}{2m_Q} + \mathcal{O}(1/m_Q^2), \quad (2)$$

where $d_P = 3$ for a pseudoscalar meson, and $d_V = -1$ for a vector meson. The parameter $\bar{\Lambda}$ corresponds to the effective mass of the light degrees of freedom in the $m_Q \rightarrow \infty$ limit [17], and the coefficients λ_1 and $\lambda_2(m_Q)$ parameterize the contributions from the kinetic and the chromo-magnetic operator [18]. They are defined as ($M = P$ or V)

$$\begin{aligned} \frac{\langle M(v) | \bar{h}_v (iD_\perp)^2 h_v | M(v) \rangle}{\langle M(v) | \bar{h}_v h_v | M(v) \rangle} &= \lambda_1 = -2m_Q K_Q, \\ C(m_Q) \frac{\langle M(v) | \bar{h}_v g_s \sigma_{\mu\nu} G^{\mu\nu} h_v | M(v) \rangle}{\langle M(v) | \bar{h}_v h_v | M(v) \rangle} &= 2d_M \lambda_2(m_Q). \end{aligned} \quad (3)$$

The quantity λ_1 is a fundamental parameter of HQET, which is independent of the heavy quark mass. We note that $K_Q = -\lambda_1/2m_Q$ is the expectation value (in the effective theory) of the kinetic energy of the heavy quark in the hadron's rest frame. It is the same for pseudoscalar and vector mesons. The second parameter, λ_2 , contains a logarithmic dependence on m_Q arising from the short-distance coefficient $C(m_Q)$.

Both λ_1 and λ_2 play a crucial role in the description of power corrections to the heavy quark limit. For instance, the leading nonperturbative corrections to inclusive semileptonic decay rates are completely determined in terms of these two quantities [11, 12, 13, 14]. An estimate of λ_2 can be extracted from the observed mass splitting between B and B^* mesons: $\lambda_2(m_b) \simeq \frac{1}{4}(m_{B^*}^2 - m_B^2) \simeq 0.12 \text{ GeV}^2$. The parameter λ_1 , however, is not directly related to an observable quantity. In the phenomenological model of Ref. [19], the kinetic energy results from the Fermi motion of the heavy quark inside the hadron, and one expects $-\lambda_1 \approx p_F^2 \approx 0.1 \text{ GeV}^2$, where p_F is the Fermi momentum. On the other hand, a recent QCD sum rule analysis predicts the much larger value $-\lambda_1 = 0.6 \pm 0.1 \text{ GeV}^2$ [20].

In this paper, we shall use the formalism of HQET to derive the field-theory analog of the virial theorem, which relates λ_1 to a matrix element of the gluon field strength tensor between hadron states of *different* velocity. We show that the existing QCD sum rule calculations of λ_1 in Refs. [20, 21, 22] are not consistent with this general relation. The results obtained from these analyses should therefore be taken with some caution. In fact, we argue that the kinetic energy is probably smaller than predicted in these approaches.

2 Derivation of the Virial Theorem

In order to derive the virial theorem, one has to study in detail the structure of hadronic matrix elements of local dimension-five current operators in HQET. Such an analysis was performed in Ref. [18], where the second-order power corrections to meson and baryon weak decay form factors have been investigated. Here we shall elaborate on some results obtained in this work.

Since the states of HQET are taken to be the eigenstates of the leading term in the effective Lagrangian (1), matrix element evaluated between these states have well-defined transformation properties under the spin-flavor symmetry group. In addition, these matrix elements are constrained by the

equations of motion of HQET, and by the requirement of Lorentz covariance. These constraints are most elegantly incorporated in the covariant tensor formalism introduced in Ref. [4]. Hadrons containing a heavy quark and light degrees of freedom are represented by tensors with the correct transformation properties under the Lorentz group and under rotations of the heavy quark spin. For instance, the doublet of the ground-state pseudoscalar and vector mesons is represented by 4×4 Dirac matrices

$$\mathcal{M}(v) = \frac{1 + \not{v}}{2\sqrt{2}} \begin{cases} -\gamma_5 & \text{; pseudoscalar meson } P(v), \\ \not{\epsilon} & \text{; vector meson } V(v, \epsilon), \end{cases} \quad (4)$$

where ϵ^μ is the polarization vector of the vector meson ($\epsilon \cdot v = 0$). We use a nonrelativistic normalization of states, such that ($M = P$ or V)

$$\langle M(v) | \bar{h}_v h_v | M(v) \rangle = -\text{Tr} \{ \bar{\mathcal{M}}(v) \mathcal{M}(v) \} = 1. \quad (5)$$

As in this example, meson matrix elements of operators in HQET can always be written as traces over these wave functions; however, for more complicated matrix elements one has to include appropriate tensors for the light degrees of freedom, too.

Consider, as an example, the matrix element of a dimension-five current operator containing the gluon field strength tensor between meson states with different velocity. Using the tensor formalism, one can write

$$\langle M(v') | \bar{h}_{v'} \Gamma i g_s G^{\mu\nu} h_v | M(v) \rangle = -\text{Tr} \{ \phi^{\mu\nu}(v, v') \bar{\mathcal{M}}(v') \Gamma \mathcal{M}(v) \} \quad (6)$$

where Γ may be an arbitrary combination of Dirac matrices. The object $\phi^{\mu\nu}(v, v')$ represents the light degrees of freedom. It is a most complicated hadronic quantity which, however, can only depend on the meson velocities. Using the projection property $\not{v} \mathcal{M}(v) = \mathcal{M}(v) = -\mathcal{M}(v) \not{v}$ of the meson wave functions, one finds that the most general decomposition of the form factor is

$$\begin{aligned} \phi^{\mu\nu}(v, v') &= \phi_1(w) (v^\mu v'^\nu - v^\nu v'^\mu) \\ &+ \phi_2(w) [(v - v')^\mu \gamma^\nu - (v - v')^\nu \gamma^\mu] + i \phi_3(w) \sigma^{\mu\nu}, \end{aligned} \quad (7)$$

where $w = v \cdot v'$. T -invariance of the strong interactions requires that the scalar functions $\phi_i(w)$ be real. Note that the matrix element must be invariant under complex conjugation accompanied by an interchange of the indices

and the velocity (and polarization) vectors. This forbids a term proportional to $[(v + v')^\mu \gamma^\nu - (v + v')^\nu \gamma^\mu]$.

Using the equations of motion of HQET, one can show that the normalization of the functions $\phi_i(w)$ at zero recoil ($w = 1$) is related to the parameters λ_1 and λ_2 . One finds [18]

$$\phi_1(1) - \phi_2(1) - \frac{1}{2} \phi_3(1) = -\frac{\lambda_1}{3}, \quad \phi_3(1) = \lambda_2. \quad (8)$$

For completeness, we give the details of the derivation of this result in the appendix. Note that the first equation relates λ_1 to form factors parameterizing the matrix elements of the operator containing the gluon field strength tensor.

In order to proceed, we evaluate (6) for $\Gamma = 1$, both for pseudoscalar and vector mesons. In the latter case the polarization vector is chosen to be the same in the initial and final state. We obtain

$$\begin{aligned} \langle P(v') | \bar{h}_{v'} i g_s G^{\mu\nu} h_v | P(v) \rangle &= \langle V(v', \epsilon) | \bar{h}_{v'} i g_s G^{\mu\nu} h_v | V(v, \epsilon) \rangle \\ &= \frac{1}{2} (v^\mu v'^\nu - v^\nu v'^\mu) \left[(w + 1) \phi_1(w) - 2\phi_2(w) - \phi_3(w) \right] \\ &= (v^\mu v'^\nu - v^\nu v'^\mu) \left[-\frac{\lambda_1}{3} + \mathcal{O}(w - 1) \right]. \end{aligned} \quad (9)$$

A corresponding relation holds for the ground-state Λ_Q baryons containing a heavy quark, too. In this case, the analogs of the functions $\phi_2(w)$ and $\phi_3(w)$ vanish, and the analog of $\phi_1(w)$ is normalized at zero recoil to $-\tilde{\lambda}_1/3$ [18]. The parameter $\tilde{\lambda}_1$ parameterizes the baryon matrix element of the kinetic operator and is defined in analogy to (3). Taking into account that both λ_1 and $\tilde{\lambda}_1$ are proportional to the kinetic energy K_Q of the heavy quark inside the hadron, we obtain the virial theorem

$$\frac{\langle H_Q(v') | \bar{h}_{v'} i g_s G^{\mu\nu} h_v | H_Q(v) \rangle}{\langle H_Q(v) | \bar{h}_v h_v | H_Q(v) \rangle} = \frac{2m_Q}{3} (v^\mu v'^\nu - v^\nu v'^\mu) \left[K_Q + \mathcal{O}(w - 1) \right], \quad (10)$$

which in this form is independent of the normalization of the states. The hadron H_Q can be any of the ground-state heavy mesons or baryons.

It is instructive to evaluate this relation in the rest frame of the initial hadron, where the nonvanishing components of the tensor on the right-hand

side are those with $\mu = 0$ or $\nu = 0$. We find

$$\frac{\langle H_Q(\vec{v}') | \bar{h}_{\vec{v}'} i g_s \vec{E}_c h_{\vec{0}} | H_Q(\vec{0}) \rangle}{\langle H_Q(\vec{0}) | \bar{h}_{\vec{0}} h_{\vec{0}} | H_Q(\vec{0}) \rangle} = -\frac{2m_Q}{3} \vec{v}' K_Q + \mathcal{O}(\vec{v}'^3), \quad (11)$$

where $E_c^i = -G^{0i}$ are the components of the chromo-electric field.

In quantum mechanics, the matrix element on the left-hand side can be represented as

$$\langle H_Q | \exp(im_H \vec{v}' \cdot \vec{\mathbf{x}}) i g_s \vec{E}_c(\vec{\mathbf{x}}) | H_Q \rangle, \quad (12)$$

where the states are at rest and normalized to unity. The final state hadron with momentum $m_H \vec{v}'$ has been related to a hadron at rest by a boost operator, where $\vec{\mathbf{x}}$ denotes the operator for the center-of-mass coordinate. In the heavy quark limit, one can identify $\vec{\mathbf{x}}$ with the position operator for the heavy quark. We can now expand the exponential in powers of \vec{v}' , using that by rotational invariance

$$\begin{aligned} \langle H_Q | \vec{E}_c(\vec{\mathbf{x}}) | H_Q \rangle &= 0, \\ \langle H_Q | \mathbf{x}^i E_c^j(\vec{\mathbf{x}}) | H_Q \rangle &= \frac{\delta^{ij}}{3} \langle H_Q | \vec{\mathbf{x}} \cdot \vec{E}_c(\vec{\mathbf{x}}) | H_Q \rangle. \end{aligned} \quad (13)$$

Equating the terms linear in \vec{v}' in (11), and using that $m_H/m_Q = 1$ to leading order in $1/m_Q$, we obtain

$$2K_Q = \langle H_Q | g_s \vec{\mathbf{x}} \cdot \vec{E}_c(\vec{\mathbf{x}}) | H_Q \rangle. \quad (14)$$

In the abelian case, where the chromo-electric field can be written in terms of a potential, and $g_s \vec{E}_c(\vec{x}) = \vec{\nabla} V(\vec{x})$ is the gradient of the potential energy of the heavy quark interacting with the background field of the light degrees of freedom, this is nothing but the classical virial theorem.

3 Conclusions

Using the formalism of the heavy quark effective theory, we have derived the field-theory version of the virial theorem, which relates the matrix element of the kinetic energy of a heavy quark inside a hadron to a matrix element of the gluon field strength tensor between hadron states with different velocity. This theorem is interesting in that it shows that the two parameters λ_1 and

λ_2 , which play an important role in the description of power corrections to the heavy quark limit, have a similar origin.

The virial theorem has direct implications for existing QCD sum rule calculations of the kinetic energy. In these analyses, the main contribution comes from a diagram not containing gluons (the bare quark loop). However, when the kinetic energy is calculated by constructing a sum rule for the left-hand side of (10), such a graph does not appear. Hence, it cannot contribute to the sum rule for λ_1 . In other words, the virial theorem makes explicit an “intrinsic smallness” of the kinetic energy, which in the existing sum rule calculations would have to result from a cancellation of several large contributions. We conclude that the numerical results for the kinetic energy quoted in Refs. [20, 21, 22] should be taken with caution. A new QCD sum rule analysis, which incorporates the virial theorem, will be presented elsewhere [23]. Since in this sum rule the leading contribution comes from a two-loop diagram, the resulting value of the kinetic energy will be considerably smaller.

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Appendix

For completeness, we give below the main steps of a derivation presented in Ref. [18], which leads to the zero recoil conditions (8). Consider the matrix element

$$\langle M(v') | (-iD^\mu \bar{h}_{v'}) \Gamma iD^\nu h_v | M(v) \rangle = -\text{Tr} \left\{ \psi^{\mu\nu}(v, v') \overline{\mathcal{M}}(v') \Gamma \mathcal{M}(v) \right\}. \quad (\text{A.1})$$

The most general decomposition of the tensor form factor $\psi^{\mu\nu}(v, v')$ involves ten invariant functions. However, considering the complex conjugate of the above matrix element, one finds the symmetry relation $\overline{\psi}^{\mu\nu}(v', v) = \psi^{\mu\nu}(v, v')$, which reduces the number of invariant functions to seven. It is convenient to perform a decomposition into symmetric and antisymmetric

parts, $\psi^{\mu\nu} = \frac{1}{2}(\psi_S^{\mu\nu} + \psi_A^{\mu\nu})$, and to define

$$\begin{aligned}\psi_S^{\mu\nu}(v, v') &= \psi_1^S(w) g^{\mu\nu} + \psi_2^S(w) (v + v')^\mu (v + v')^\nu \\ &\quad + \psi_3^S(w) (v - v')^\mu (v - v')^\nu + \psi_4^S(w) \left[(v + v')^\mu \gamma^\nu + (v + v')^\nu \gamma^\mu \right],\end{aligned}\tag{A.2}$$

$$\begin{aligned}\psi_A^{\mu\nu}(v, v') &= \psi_1^A(w) (v^\mu v'^\nu - v'^\mu v^\nu) \\ &\quad + \psi_2^A(w) \left[(v - v')^\mu \gamma^\nu - (v - v')^\nu \gamma^\mu \right] + i\psi_3^A(w) \sigma^{\mu\nu},\end{aligned}$$

where $w = v \cdot v'$. Because of T -invariance of the strong interactions, the invariant functions are real.

The equations of motion of HQET imply that, under the trace in (A.1), $v_\nu \psi^{\mu\nu}(v, v') \doteq 0$. This leads three relations among the seven functions, which can be used to eliminate $\psi_1^S(w)$, $\psi_2^S(w)$, and $\psi_4^S(w)$.

An integration by parts relates (A.1) to a matrix element of an operator containing two derivatives acting on the same heavy quark field. The result is

$$\begin{aligned}\langle M(v') | \bar{h}_{v'} \Gamma iD^\mu iD^\nu h_v | M(v) \rangle &= \langle M(v') | (-iD^\mu \bar{h}_{v'}) \Gamma iD^\nu h_v | M(v) \rangle \\ &\quad + \bar{\Lambda} (v - v')^\mu \langle M(v') | \bar{h}_{v'} \Gamma iD^\nu h_v | M(v) \rangle,\end{aligned}\tag{A.3}$$

where $\bar{\Lambda} = M - m_Q$. The second matrix element on the right-hand side can be written as [8]

$$\langle M(v') | \bar{h}_{v'} \Gamma iD^\nu h_v | M(v) \rangle = -\text{Tr} \left\{ \xi^\nu(v, v') \overline{\mathcal{M}}(v') \Gamma \mathcal{M}(v) \right\}, \tag{A.4}$$

where

$$(w+1) \xi^\nu(v, v') = (w v^\nu - v'^\nu) \bar{\Lambda} \xi(w) - \left[(v + v')^\nu + (w+1) \gamma^\nu \right] \xi_3(w). \tag{A.5}$$

One can use the above results to relate the functions $\phi_i(w)$ defined in (7), as well as the function $\phi_0(w)$ defined as

$$\langle M(v') | \bar{h}_{v'} \Gamma (iD_\perp)^2 h_v | M(v) \rangle = -\phi_0(w) \text{Tr} \left\{ \overline{\mathcal{M}}(v') \Gamma \mathcal{M}(v) \right\}, \tag{A.6}$$

to the four functions $\psi_3^S(w)$ and $\psi_i^A(w)$. The result is

$$\begin{aligned}
\phi_0(w) &= -\frac{2w+1}{w+1} \left[(w+1) \psi_1^A(w) - 2\psi_2^A(w) - \psi_3^A(w) \right] \\
&\quad + 2(w-1) \psi_3^S(w) - \bar{\Lambda}^2(w-1) \xi(w), \\
\phi_1(w) &= \psi_1^A(w) + \frac{1}{w+1} \left[\bar{\Lambda}^2(w-1) \xi(w) - 2\bar{\Lambda} \xi_3(w) \right], \\
\phi_2(w) &= \psi_2^A(w) - \bar{\Lambda} \xi_3(w), \\
\phi_3(w) &= \psi_3^A(w).
\end{aligned} \tag{A.7}$$

From a comparison with (3), one concludes that $\phi_0(1) = \lambda_1$ and $\phi_3(1) = \lambda_2$. This leads to the relations given in (8).

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